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Structural Properties of a $Z(N^2)$-Spin Model and Its Equivalent $Z(N)$-Vertex Model

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We show that a $Z(N^2)$-spin model proposed by A. B. Zamolodchikov and M. I. Monastyrskii can be conveniently described by two interacting N-state Potts models. We study its properties, especially by using a dual invariant quantity of the model. A partial duality performed on one set of Potts spins yields a staggered $Z(N)$-symmetric vertex model, which turns out to be a generalization of the N-state "nonintersecting string model" of C. L. Schultz and J. H. H. Perk. We describe its properties and elaborate on its (pseudo) "weak-graph symmetry." As by-products we find alternative representations of the N^2-state and N-state Potts models by staggered Schultz–Perk vertex models, as compared to the usual representation by staggered six-vertex models.

KEY WORDS: Two-dimensional spin and vertex systems; $Z(N)$ symmetry; duality, partial-duality, weak-graph symmetry; integrability.

1. INTRODUCTION

The understanding of critical phenomena is greatly helped by the existence of statistical models in two dimensions which happen to be exactly solvable. They are, for example, the Ising model, the six-vertex model, and the eight-vertex model, respectively, solved by Onsager, Lieb, and Baxter. The three models exhibit very different critical behavior, although they are related to each other. They also served as prototypes for two-component systems, either on sites or on bonds of a lattice.

Generalizations of the Ising, six-vertex, and eight-vertex models exist at hand. The most popular generalization of the Ising model is the N-state Potts model, involving only one interaction parameter between neigh-

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boring sites. Although it has been the object of extensive studies since it was introduced,[1] it can be only solved in special cases.[2] The "vector" version of the N-state Potts model, known as the "clock model" or $Z(N)$-spin model, because the spin values lie on the unit circle, in its most general form is of recent origin[3]; it depends on $N/2$ [resp. $(N-1)1/2$] coupling parameters for N even (resp. N odd). Little is known on this $Z(N)$-spin model except some symmetries and a duality relation. On the other hand extensions of the six-vertex model to N states per bond, in particular soluble ones, have been found by many authors.[4] Recently a generalization of the eight-vertex model has been proposed by Belavin[5] [hereafter called the $Z(N)$-Belavin model] which possesses an integrable case depending on three parameters, and has been studied rather extensively in the literature.[6] It is not known, however, whether the $Z(N)$-Belavin model is related to a $Z(N)$-spin model; one possible reason is the absence of some symmetries inherent to the eight-vertex model.

A remarkable connection between spin and vertex systems is realized in the so-called Ashkin–Teller model.[7] Originally this model is meant to be a four-state generalization of the Ising model, having also a duality relation. It was shown subsequently that it may be represented by two overlapping Ising models interacting through a bond–bond interaction.[8] A partial-duality transformation on one set of Ising spins[9] leads to its equivalence to a staggered eight-vertex systems, which becomes soluble in some special instances.[10] It is therefore natural to raise the question whether there exists an N-state generalization of the Ising model which would display a similar connection to some N-state generalization of the eight-vertex model. In this paper we show that this question has a concrete answer.

The starting point of our work is a model, called P_{NN}, which has been introduced by Zamolodchikov and Monastyrskii in 1979 in their discussion on duality relations for generalized spin systems.[11] The P_{NN} model seems to us an appropriate choice for study since it does concern spin having more than two values and yet controlled by only two coupling constants, which could be responsible for multicriticality and nonuniversal behavior.

We begin by showing that the P_{NN} model consists of two N-state Potts models lying on top of each other and coupled by a bond–bond interaction generalizing thus the Ashkin–Teller mechanism to Potts spins. In 1979 Domany and Riedel[12] have also introduced such models under the name of (N_α, N_β) models, which they analyzed by global duality and renormalization group methods. In particular, they discussed a number of special cases: $(N_\alpha, 1)$, $(N_\alpha, 2)$, $(2, 2)$, $(3, 2)$, and $(4, 2)$ and gave their physical interpretation. In fact, these models turn out to be the $P_{N_1 N_2}$ models of Ref. 11. For the sake of completeness, let us also mention that there exists another generalization of the Ashkin–Teller model[13,14] consisting of piling M-Ising spin families, coupled pairwise, which has been studied by mean field theory and Monte Carlo methods and recently solved in the limit $M \to \infty$.[15] Such a construction for N-state Potts spins has just appeared and also shown to be soluble for $N \to \infty$.[16]

In Section 3 we discuss the main symmetry properties of an isotropic P_{NN} model. A brief derivation of the duality relations obtained by Zamolodchikov and Monastyrskii is presented following the prescription of Wu and Wang.[17] We then show that the duality transformation may be simply represented by a reflection in a new set of variables obtained by a birational transformation from the original ones. We construct then dual invariants $\Delta_{Nj}(j=h, v)$ associated to both horizontal and vertical bonds and observe that in an isotropic P_{NN} model, $\Delta_{Nj}=\Delta_N$ may serve to parametrize a family of thermodynamical paths connecting the low-temperature regime to the high-temperature regime, which remain invariant under duality. We calculate also the various transformations of the weights connected to the so-called "inverse relations" of the partition function and comment on their behavior.

The transformation of partial duality on one set of Potts spins is performed in the next section. The resulting spin system is then shown to be equivalent to a staggered vertex system with $Z(N)$ spins on bonds of the medial lattice. We may think of our $Z(N)$ vertex model as a generalization of the eight-vertex model. This is however not the $Z(N)$-Belavin vertex model but it appears to be the generalization of the N-state "nonintersecting string vertex model" of Schultz and Perk.[18,19] This is the main result of this paper. More precisely we want to point out that the vertex system just obtained is a subclass of a more general class of $Z(N)$-symmetric vertex models: the generalization of the eight-vertex model is certainly not unique.

In Section 5 we study this new $Z(N)$-vertex system for its own sake. We give a complete account of the symmetries of the partition function with respect to the N^2 weights defining the system. The main part of the discussion is devoted to establishing a (pseudo) weak-graph symmetry which generalized that of the eight-vertex model. Furthermore, the restricted $Z(N)$-vertex system describing the P_{NN} model may be viewed as the generalization of the special (in fact critical) eight-vertex model associated to the symmetric Ashkin–Teller model.[9] It is thus possible to reduce it via a weak-graph symmetry transformation to a Schultz–Perk model, in the way a critical eight-vertex is brought to the form of a six-vertex model.[20] Then the dual invariant introduced earlier in Section 3 appears as the natural generalization of Lieb's invariant of the six-vertex model. Following the results of Schultz and Perk[19] one may already infer when the subclass of $Z(N)$-vertex system is integrable; we have not however searched for the solubility conditions of the general case.

Specializing now to $\Delta_{Nj} = -1$ ($j = h, v$) we obtain the following result: the N^2-state Potts model is equivalent to a staggered Schultz–Perk vertex system, which may be equivalently described by N-color nonintersecting polygon contours on a square lattice.[19] This is an alternative representation to the known staggered (two-color) polygon contours of Wu,[21] itself equivalent to a staggered ice rule. At criticality, in the ferromagnetic regime, the staggering effect vanishes and the Schultz–Perk system has the same free energy as a six-vertex system. This coincidence, which was observed some time ago[18] and analyzed recently with the Bethe–ansatz wave function,[19] is in no way accidental but can be fully explained via the equivalence to the N^2-state Potts model.

In the last section we consider a generalization of the P_{NN} model by breaking the $Z(N^2)$ symmetry down to $Z(N) \otimes Z(N)$. We discuss then the equivalent vertex models in some special limits. In one instance we find the generalization of Wu's representation of the Ising model by staggered ice rule having only two nonvanishing weights: here the N-state Potts model is represented by a staggered system of "N-color corners" only with two nonvanishing weights.[22] We conclude this study by giving a short account of what has been accomplished and the listing of some unsolved problems still open for investigation.

2. THE P_{NN} MODEL

Zamolodchikov and Monastyrskii originally constructed the P_{NN} model by taking a spin space consisting of N^2 elements: s_i^a, labeled by $i = 1, 2,..., N$ and $a = 1, 2,..., N$ and by defining an interaction between neighboring sites of a square lattice, through a Hamiltonian $H(s_i^a(n), s_j^b(m))$ displaying a maximal symmetry $Z(N^2)$:

$$\beta H(s_i^a(n), s_j^b(m)) = \{\beta_2 + (\beta_1 - \beta_2)\delta_{ij} - \beta_1 \delta_{ij}\delta_{ab}\} \tag{1}$$

where $\beta = (kT)^{-1}$, $\beta_{1,2} = \varepsilon_{1,2}(kT)^{-1}$ and n, m are two neighboring sites.

To recast the system in a more familiar setting[17] we compute the so-called U matrix associated to sites n and m, whose matrix elements are $\exp \beta H(s_i^a(n), s_j^b(m))$. The symmetry $Z(N^2)$ is most appropriately taken care of by using the matrix representation of the $Z(N^2)$-generalized Clifford algebra, which is of recent origin in the mathematical literature.[23] For our purpose we shall consider the particular representation with two generators Λ and R defined by

$$\Lambda = \begin{pmatrix} 1 & & & 0 \\ & \zeta & & \\ & & \ddots & \\ 0 & & & \zeta^{N^2-1} \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix} \tag{2}$$

with $\zeta = \exp(2i\pi/N^2)$ and the properties

$$\Lambda^{N^2} = R^{N^2} = \mathbb{1}_{N^2}, \qquad R\Lambda = \zeta \Lambda R \tag{3}$$

Inspection shows that the U matrix of the P_{NN} model is of the form

$$U = \mathbb{1}_{N^2} + \omega \sum_{j=0}^{N-1} \sum_{l=1}^{N-1} R^{jN+l} + \omega'' \sum_{j=1}^{N-1} R^{jN} \tag{4}$$

where $\omega = e^{-\beta_2}$ and $\omega'' = e^{-\beta_1}$.

We now show that this U matrix can be alternatively obtained by taking two sets of classical spins at a given site n, i.e., $\{S_n\}$ and $\{S'_n\}$ taking the values: $1, \xi,..., \xi^{N-1}$ with $\xi = \zeta^N$ and constructing the following interaction:

$$\beta H_{nm} = \{K_0 + KB(S_n, S_m) + K''B(S'_n, S'_m) + (N-1)\, KB(S_n, S_m)\, B(S'_n, S'_m)\} \tag{5}$$

where K and K'' are coupling parameters and $B(S_n, S_m)$ is given by

$$\begin{aligned} B(S_n, S_m) &= (N-1)^{-1} \sum_{j=1}^{N-1} S_n^j S_m^{N-j} \\ &= \{\delta(S_n - S_m) - (N-1)^{-1}\} \end{aligned} \tag{6}$$

By computing $\exp \beta H_{nm}$ and choosing $K_0 = -(NK + K')$ we see that we reobtain precisely Eq. (4) with

$$\omega'' = \exp\left\{-\frac{N^2}{(N-1)} K\right\} \qquad \text{and} \qquad \omega = \exp\left\{-NK - \frac{N}{(N-1)} K''\right\} \tag{7}$$

Since $\exp[K(N-1)^{-1} + KB(S_n, S_m)]$ describes the matrix elements of a U matrix of an N-state Potts model,[17] the P_{NN} model may be viewed as two N-state Potts models lying on top of each other and interacting through a bond–bond type of interaction, i.e., $B(S_n, S_m)\, B(S'_n, S'_m)$, which generalizes the symmetric Ashkin–Teller construction to $Z(N)$ spins.

Then the partition function $\mathscr{Z}_{MM'}$ for a lattice of M columns and M'

rows with toroidal boundary conditions, of an anisotropic P_{NN} model, is (h, horizontal; v, vertical)

$$\mathscr{Z}_{MM'}(\omega_v, \omega''_v; \omega_h, \omega''_h) = \sum_{\{S_n\}\{S'_n\}} \prod_{\langle n,m\rangle_v} \exp \beta H^v_{nm} \prod_{\langle n,m\rangle_h} \exp \beta H^h_{nm} \tag{8}$$

which can be expressed as the M'th power of a transfer matrix $T(M)$:

$$T(M) = T_2(M)\, T_1(M) \tag{9}$$

$$\mathscr{Z}_{MM'}(\omega_v, \omega''_v; \omega_h, \omega''_h) = \mathrm{Tr}\{T(M)\}^{M'} \tag{10}$$

where $T_1(M)$ and $T_2(M)$ are given in terms of A_m and R_m, the matrices pf Eq. (2) at site m on a given row by

$$\begin{aligned} T_1(M) = \exp \sum_{m=1}^{M} \Bigg[\frac{K_h}{(N-1)} \sum_{j=0}^{N-1} \sum_{l=1}^{N-1} A_m^{jN+l}(A_{m+1}^{jN+l})^+ \\ + \frac{K''_h}{(N-1)} \sum_{j=1}^{N-1} A_m^{jN}(A_{m+1}^{jN})^+ + K_0 \Bigg] \end{aligned} \tag{11}$$

$$T_2(M) = \prod_{m=1}^{M} \left\{ \mathbb{1}_{N^2} + \omega_v \sum_{j=0}^{N-1} \sum_{l=1}^{N-1} R_m^{jN+l} + \omega''_v \sum_{j=1}^{N-1} R_m^{jN} \right\}$$

with ω_j, ω''_j ($j = h, v$) related to K_j and K''_j by Eq. (7).

3. PROPERTIES OF THE PARTITION FUNCTION

First, we quote a trivial symmetry S which consists in rotating the lattice by 90°:

$$S\colon (\omega_v, \omega''_v; \omega_h, \omega''_h) \to (\omega_h, \omega''_h; \omega_v, \omega''_v) \tag{12}$$

$$\mathscr{Z}_{MM'}(\omega_v, \omega''_v; \omega_h, \omega''_h) = \mathscr{Z}_{M'M}(\omega_h, \omega''_h; \omega_v, \omega''_v) \tag{13}$$

Let us divide now the square lattice into two interpenetrating sublattices such that a site of one sublattice is surrounded by the four sites of the other sublattice. For N even, the values $1, \xi, ..., \xi^{N-1}$ taken by a classical $Z(N)$ spin S_n at site n are symmetric with respect to sign reversal. Thus it makes sense to speak of the substitution $S_n \to -S_n$ on one sublattice; changing simultaneously then $K_j \to -K_j$ but keeping all S'_n spins and K''_j their coupling K''_j fixed we see that Eq. (5) remains invariant. This fact can be written out as a transformation T for N even and $j = v, h$:

$$\begin{aligned} T\colon \omega''_j &\to (\omega''_j)^{-1} = \omega''^{T}_j \\ \omega_j &\to \omega_j(\omega''_j)^{-2(N-1)/N} = \omega^T_j \end{aligned}$$

$$\mathscr{Z}_{MM'}(\omega_v, \omega''_v; \omega_h, \omega''_h) = \mathscr{Z}_{MM'}(\omega^T_v, \omega''^T_v; \omega^T_h \omega''^T_h) \tag{14}$$

An invariant combination of weights under T is

$$\frac{\omega_j^{N/(N-1)}}{\omega_j''} = \exp -\frac{N^2}{(N-1)^2} K_j'' \tag{15}$$

For $N=2$, there is even a higher symmetry due to the fact that a product of Ising spins is again an Ising spin.

For N odd, T does not exist since it maps the system into a new spin system which takes values in the set of Nth roots of (-1).

Most spin systems display a duality symmetry, a property which states that the partition function can be alternatively, up to trivial factors expressed as the partition function of another set of spins defined on the dual lattice (in our case the dual lattice remains a square lattice) with appropriate weights, rational functions of the original weights. The functional form of $\mathscr{Z}_{MM'}$ remains thus the same for a specific substitution of the weights found already by Zamolodchikov and Monastyrskii for the P_{NN} model. Mathematically, it is a kind of Fourier representation of $\mathscr{Z}_{MM'}$ first observed by McKean for the Ising model later on generalized to several groups by Zamolodchikov and Monastyrskii.[11]

Here we take advantage of the simplicity of the pevious matrix representation and calculate explicitly the dual weights: ω_j^*, $\omega_j''^*$, $j=v, h$. Our starting point is a relation derived by Wu and Wang[17] which states that the dual weights are obtained by the non-normalized eigenvalues of the matrix U.

The diagonalization of U can be easily done by using a $N^2 \times N^2$ matrix $\mathbf{D}$ defined by[1]

$$\mathbf{D} = N^{-1}\zeta^{ij}, \qquad i \text{ and } j = 0, 1, 2,..., N^2-1 \tag{16}$$

which has the property of turning R into Λ:

$$\mathbf{D}^{-1} R \mathbf{D} = \Lambda \tag{17}$$

The transform of U is diagonal of the form

$$\mathbf{D}^{-1} U \mathbf{D} = \mathbb{1}_{N^2} + \omega \sum_{j=0}^{N-1} \sum_{k=1}^{N-1} \Lambda^{jN+k} + \omega'' \sum_{j=1}^{N-1} \Lambda^{jN} \tag{18}$$

We find the following eigenvalues:

$$\begin{aligned}
&\lambda_0 = 1 + (N-1)\omega'' + N(N-1)\omega \\
&\lambda_N = \lambda_{2N} = \cdots = \lambda_{N(N-1)} = 1 + (N-1)\omega'' - N\omega \\
&\lambda_l = 1 - \omega'', \qquad l \neq 0, N, 2N,..., N(N-1)
\end{aligned} \tag{19}$$

Hence the duality relations for the weights are

$$\omega^*_{v,h} = \frac{1-\omega''_{h,v}}{1+(N-1)\omega''_{h,v}+N(N-1)\omega_{h,v}},$$
$$\omega^{*''}_{v,h} = \frac{1+(N-1)\omega''_{h,v}-N\omega_{h,v}}{1+(N-1)\omega''_{h,v}+N(N-1)\omega_{h,v}} \tag{20}$$

and for the partition function

$$\mathscr{Z}_{MM'}(\omega_v, \omega''_v; \omega_h, \omega''_h) = A_{MM'}\mathscr{Z}_{MM'}(\omega^*_v, \omega^{*''}_v; \omega^*_h, \omega^{*''}_h) \tag{21}$$

where

$$A_{MM'} = N^{2(1-MM')}\{1+(N-1)\omega''_v+N(N-1)\omega_v\}^{MM'} \times \{1+(N-1)\omega''_h+(N-1)N\omega_h\}^{MM'}$$

The linear birational transformation defined by Eq. (20) can be reduced to a reflection by the following choice of variables:

$$\mathbf{a}_{v,h} = \frac{N\omega_{v,h}}{1+(N-1)\omega''_{v,h}}, \qquad \mathbf{b}_{v,h} = \frac{1-\omega''_{v,h}}{1+(N-1)\omega''_{v,h}} \tag{22}$$

or conversely

$$\omega_{v,h} = \frac{\mathbf{a}_{v,h}}{1+(N-1)\mathbf{b}_{v,h}}, \qquad \omega''_{v,h} = \frac{1-\mathbf{b}_{v,h}}{1+(N-1)\mathbf{b}_{v,h}} \tag{23}$$

Then Eq. (20) can be recast as

$$\omega^*_{v,h} = \frac{\mathbf{b}_{h,v}}{1+(N-1)a_{h,v}}, \qquad \omega^{*''}_{v,h} = \frac{1-\mathbf{a}_{h,v}}{1+(N-1)\mathbf{a}_{h,v}} \tag{24}$$

Now for dual weights ω^*_j and ω''^*_j there corresponds according to Eq. (24) $\mathbf{a}^*_j$ and $\mathbf{b}^*_j$ which are simply related to $\mathbf{a}_j$ and $\mathbf{b}_j$ by

$$\mathbf{a}^*_{v,h} = \frac{N\omega^*_{v,h}}{1+(N-1)\omega''^*_{v,h}} = \frac{N\mathbf{b}_{h,v}}{1+(N-1)\mathbf{a}_{h,v}+(N-1)-(N-1)\mathbf{a}_{h,v}} = \mathbf{b}_{h,v} \tag{25}$$

$$\mathbf{b}^*_{v,h} = \frac{1-\omega''^*_{v,h}}{1+(N-1)\omega''^*_{v,h}} = \frac{1+(N-1)\mathbf{a}_{h,v}-1+\mathbf{a}_{h,v}}{1+(N-1)\mathbf{a}_{h,v}+(N-1)-(N-1)\mathbf{a}_{h,v}} = \mathbf{a}_{h,v} \tag{26}$$

Equations (25), (26) show that the duality transformation is a simple reflection in these new variables; we shall see in the next section the physical meaning of the new weights $\mathbf{a}_j$ and $\mathbf{b}_j$ ($j = v, h$).

The duality transformation D for the weights of Eq. (20) have furthermore a remarkable representation in a new set of variables: $\Delta_{N,j}$, x_j ($j = v, h$) defined by

$$\Delta_{N,j} = \frac{(N-1)\omega_j - (N-2)\omega_j'' - \omega_j''/\omega_j}{1-\omega_j''}, \qquad x_j = \frac{1-\omega_j''}{N\omega_j} \tag{27}$$

Then

$$D: (\Delta_{N,v}, x_v; \Delta_{N,h}, x_h) \rightarrow (\Delta_{N,h}, x_h^{-1}; \Delta_{N,v}, x_v^{-1}) \tag{28}$$

This is a simple exchange of directions $h \leftrightarrow v$, whereby both Δ_{N_j} keep their functional form and x is replaced by its inverse. For a practical purpose one may assume $\Delta_{N,v} = \Delta_{N,h} = \Delta_N$ to have a "partial" anisotropic system for which the self-duality condition takes the form:

$$x_h x_v = 1 \tag{29}$$

On the manifold $\Delta_N = \text{const}$, the D transform has the usual form of the Potts model.[17] Again the meaning of $\Delta_{N,j}$ and x_j will become clear in the next section. Let us quote only that they can be expressed in a very suggestive form as

$$\Delta_{N,j} = (N-1)\frac{\mathbf{a}_j^2 + \mathbf{b}_j^2 - 1}{N\mathbf{a}_j\mathbf{b}_j} - (N-2)\frac{(\mathbf{a}_j - 1)(\mathbf{b}_j - 1)}{N\mathbf{a}_j\mathbf{b}_j}, \qquad x_j = \frac{\mathbf{b}_j}{\mathbf{a}_j} \tag{30}$$

To illustrate the role of the parameter $\Delta_{N,j}$ we consider the plane (ω, ω'') for an isotropic P_{NN} model, as shown by Fig. 1.[11] The self-dual line of equation

$$N\omega + \omega'' = 1 \tag{31}$$

is believed to be the locus of phase transition points, at least partly.

In general $\Delta_N = \text{const}$ corresponds to a second-order curve going through $(0, 0)$ and $(1, 1)$, hence may be best regarded as a thermodynamical path invariant under duality. Δ_N labels, however, some special cases of the spin system:

$\Delta_N = -\infty$ is represented by $\omega'' = 1$ or $\omega = 0$, these lines describe N-state Potts models with their respective critical points I_1 and I_2:

$$I_1\left(\omega_c = \frac{1}{\sqrt{N}+1}, \omega_c'' = 1\right) \quad \textit{and} \quad I_2\left(\omega_c = 0, \omega_c'' = \frac{1}{\sqrt{N}+1}\right)$$

If N is even we known that, because of the T symmetry from I_2 starts a critical line horizontally, the slope of its dual line at I_1 can be found by D.

$\Delta_N = -1$ is the line of the N^2-state Potts model ($\omega = \omega''$) which has a critical point on the self-dual line I_3 ($\omega_t = \omega_t'' = 1/(N+1)$). It is expected[11] that the two critical lines from I_1 and I_2 willl meet at I_3.

$\Delta_N = N-1$ is the pair of straight lines:

$$\omega = 1, \qquad \omega'' = -(N-1)\omega$$

There is another curve with $\Delta_N = 0$ intersecting the self-dual-line at $\omega_0 = \{1 + [N(N-1)]^{1/2}\}^{-1}$ (in fact a trajectory $\Delta_N = \text{const}$ meets the self-dual-line at $\omega_0 = \{1 + [N(N-1) - N\Delta_N]^{1/2}\}^{-1}$). For $N=2$ we know that the two Ising spin models making up the Askhin–Teller model will decouple at $\Delta_2 = 0$ and one may consider it as a vector-Potts model of four states, hence it is soluble. But when $N \neq 2$, there is no decoupling since

$$\omega'' = \frac{(N-1)\omega^2}{(N-2)\omega + 1} \tag{32}$$

and this is not the path of the N^2-vector Potts model. We do not know yet whether the system is integrable along this line $\Delta_N = 0$.

Finally as in the Ashkin–Teller model, there is a "duality" limit beyond which the dual weights becomes negative

$$1 + (N-1)\omega'' = N\omega \tag{33}$$

Equation (7) shows that for $K'' = 0$ we have the curve

$$\omega'' = (\omega)^{N/N-1} \tag{34}$$

which is obviously not invariant under duality, but it coincides with the condition $\Delta_N = 0$ for $N = 2$. This curve is tangent to the ω axis at $\omega'' = \omega = 0$ and to the duality envelope Eq. (33) at $\omega = \omega'' = 1$. Hence its dual transform will be also tangent to $\omega'' = 0$ and the duality envelope, and is comparable to the vector 5-state Potts model line in the phase diagram of a $Z(5)$ model.[24]

For $N > 2$ the system is not decoupled as for $N = 2$. The bond weight is now according to Eq. (5):

$$\exp\{K + (N-1)\, KB(S_n' S_m')\, B(S_n S_m)\} \tag{35}$$

Since $B(S_n' S_m')$ takes only two values 1 and $\{-(N-1)^{-1}\}$ we have an N-state Potts model with the following random coupling:

$$\begin{aligned} &\exp NKB(S_n S_m) && \text{if} \quad S_n' = S_m' \\ &1 && \text{if} \quad S_n' \neq S_m' \end{aligned} \tag{36}$$

Thus in the partition function we can think of the summation over $\{S'_n\}$ as a sum over all possible distributions of coupling constants with values NK or 0 on all the links of the lattice. The sum over $\{S_n\}$ is then performed for each of such configurations of couplings. Note that the intersection of this curve with the self-dual line has an abscissa ω_1 between $\{1+[N(N-1)]^{1/2}\} < \omega_1 < N^{-1}$.

Besides the symmetries just studied, the partition function of spin systems exhibits also "automorphic" properties, which result from the existence of the inverse transfer matrix. Since $T = T_2 T_1$ as shown in Section 2, the inverse of T_1, which is diagonal in the representation of Eq. (11) is simply

$$(T_1)^{-1} = T_1(\omega_h^{-1}, \omega_h^{-1''}) \tag{37}$$

T_2 is not diagonal but a product of local transfer matrices U which can be readily inverted. It is, however, much simpler to observe the following.[25] Let J be the transform of the pair (ω, ω'') of weights:

$$J\colon (\omega, \omega'') \to (\omega^{-1}, \omega''^{-1}) \tag{38}$$

then the inverse transform $I_h = J$ is

$$I_h = (\omega_h, \omega''_h) \to (\boldsymbol{\omega}_h = \omega_h^{-1}; \boldsymbol{\omega}''_h = \omega_h^{-1''}) \tag{39}$$

In order to calculate I_v we calculate the effect of the sequence of transformations D, J, D on the vertical weights ω_v and ω''_v:

$$I_v\colon (\omega_v` \omega''_v) \to (\boldsymbol{\omega}_v, \boldsymbol{\omega}''_v) \tag{40}$$

where the inverse weights $\boldsymbol{\omega}_v$ and $\boldsymbol{\omega}''_v$ are

$$\boldsymbol{\omega}_v = \omega_v \frac{(\omega''_v - 1)}{[1+(N-1)\omega''_v - N\omega_v][1+(N-2)\omega''_v + (N-1)^2\omega_v] + \omega_v(1-\omega''_v)(N-1)} \tag{41}$$

$$\boldsymbol{\omega}''_v = \frac{N(N-1)\omega_v^2 - \omega''_v - (N-1)\omega_v''^2 - N(N-2)\omega_v\omega''_v}{[1+(N-1)\omega''_v - N\omega_v][1+(N-2)\omega''_v + (N-1)^2\omega_v] + \omega_v(1-\omega''_v)(N-1)} \tag{42}$$

Then the partition function obeys the "inverse relation"[26]:

$$\begin{aligned}
&\mathscr{Z}_{MM'}(\omega_h\omega''_h; \omega_v, \omega''_v)\, \mathscr{Z}_{MM'}(\boldsymbol{\omega}_h\boldsymbol{\omega}''_h; \boldsymbol{\omega}_v\omega''_v) \\
&\quad = \{1 + N(N-1)\omega_v\boldsymbol{\omega}_v + (N-1)\omega''_v\omega''_v\}^{MM'}
\end{aligned}$$

The set of involutions D, S, I (made up of I_h and I_v) and T for $N=\text{even}$, generates the so-called "automorphy group" of the model, a concept newly introduced in Ref. 25 which turns out to be of interest for most spin systems. We now reformulate the inverse transform in the variable $\Delta_{N,j}$ and x_j. But let us first do it for J; from the definitions

$$x=\frac{1-\omega''}{N\omega}, \qquad \Delta_N=\frac{(N-1)\omega-(N-2)\omega''-\omega''/\omega}{1-\omega''} \tag{43}$$

We can eliminate ω^{-1} to find the equation obeyed by $(\omega''/\omega-1)$, which we call $f(x, \Delta_N)$:

$$f^2(x, \Delta_N)+(1+x)\,Nf(x, \Delta_N)+Nx(1+\Delta)=0$$

In the physical regime where both ω and ω'' are positive we should take

$$f_+(x, \Delta_N)=\left(\frac{\omega''}{\omega}-1\right)=\frac{1}{2}\{-N(1+x)+[N^2(1+x)^2-4Nx(1+\Delta_N)^{1/2}\} \tag{44}$$

as long as $\Delta_N< -1$.

Then

$$J\colon (x, \Delta_N)\to(\mathbf{x}, \mathbf{\Delta}_N) \tag{45}$$

with

$$\mathbf{x}^{-1}=N\Delta_N-x^{-1}+(N-2)x^{-1}f(x, \Delta_N) \tag{46}$$

$$\begin{aligned}\mathbf{\Delta}_N&=\Delta_N-\frac{N-2}{Nx}\{xNf(x, \Delta_N)-N(1+x)\,f(x, \Delta_N)-Nx(1+\Delta_N)\}\\&=\Delta_N+(N-2)\{x^{-1}f(x, \Delta_N)+(1+\Delta_N)\}\end{aligned} \tag{47}$$

Now using a scaling relation for $f(x, \Delta_N)$, deduced from Eq. (44)

$$f(x^{-1}, \Delta_N)=x^{-1}f(x, \Delta_N) \tag{48}$$

we note that both $\mathbf{x}$ and $\mathbf{\Delta}_N$ are expressible in terms of x^{-1} and Δ_N:

$$\mathbf{\Delta}_N=\Delta_N+(N-2)\{f(x^{-1}, \Delta_N)+(1+\Delta_N)\} \tag{49}$$

$$\mathbf{x}-1=(N\Delta_N-x^{-1})+(N-2)\,f(x^{-1}, \Delta_N) \tag{50}$$

For $N>2$, the transformation is then not algebraic, except for $\Delta_N=-1$ and $\Delta_N=N-1$ which corresponds to straight-line thermodynamical paths in Fig. 1. We can now express the inverse transform as follows:

$$I\colon \begin{pmatrix}\Delta_{N,h}; x_h\\ \Delta_{N,v} x_v\end{pmatrix}\to\begin{pmatrix}\mathbf{\Delta}_{N,h}; \mathbf{x}_h\\ \mathbf{\Delta}_{N,v}; \mathbf{x}_v\end{pmatrix} \qquad \begin{matrix}(39\text{a})\\(40\text{a})\end{matrix}$$

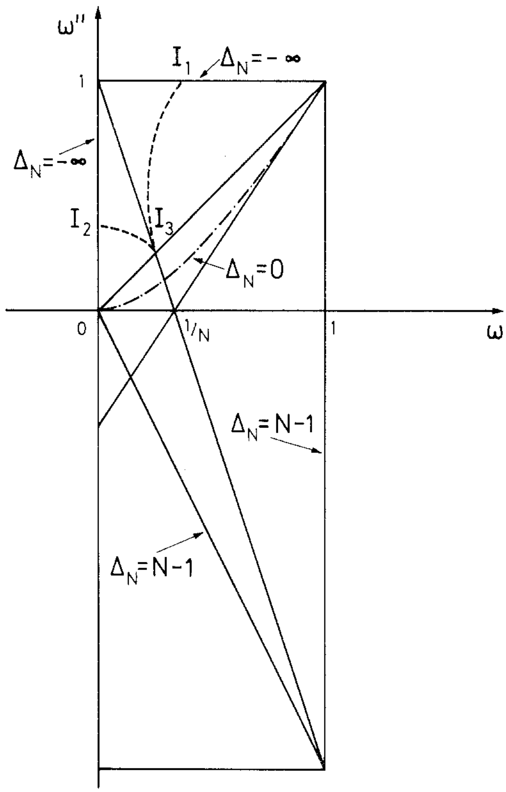


Fig. 1. Expected phase diagram of an isotropic P_{NN} model.

whereby

$$\mathbf{x}_h^{-1} = (N\Delta_{N,h} - x_h^{-1}) + (N-2)\, f(x_h^{-1}, \Delta_{N,h}) \tag{50a}$$

$$\boldsymbol{\Delta}_{N,h} = \Delta_{N,h} + (N-2)[f(x_h^{-1}, \Delta_{N,h}) + (1 + \Delta_{N,h})] \tag{49a}$$

and in the vertical direction

$$\mathbf{x}_v = (N\Delta_{N,v} - x_v) + (N-2)\, f(x_v, \Delta_{N,v}) \tag{50b}$$

$$\boldsymbol{\Delta}_{N,v} = \Delta_{N,v} + (N-2)[f(x_v, \Delta_{N,v}) + (1 + \Delta_{N,v})] \tag{49b}$$

We have taken into account the simple behavior of $\Delta_{N,j}$ and x_j under the duality transform. Although this structure is simple, the nonalgebraic part due to the function f renders the computation of the group elements of the automorphy group cumbersome. This would be done elsewhere.

Let us only observe at self-duality:

$$\Delta_{N,h} = \Delta_{N,v} = \Delta_N \qquad \text{and} \qquad x_v x_h = 1$$

the system of Eq. (50a), (49a) is the same as Eq. (49b), (50b). The case $N=2$ (symmetric Ashkin–Teller model) has been treated in Ref. 27. The inverse transform I leaves thus the self-dual manifold globally invariant.

4. EQUIVALENCE TO A STAGGERED VERTEX MODEL

In this section we perform a partial duality on one set of N-state Potts variables to construct an equivalent vertex model. The idea has been applied to the Ashkin–Teller model by Wegner in 1974.[9] To carry out the demonstration, we shall follow basically the approach of Baxter[20] and make use of the duality relation for the N-state Potts model of Wu and Wang. For the sake of simplicity we consider the isotropic P_{NN} model here.

From the previous section we know that the partition function is written as

$$\begin{aligned}\mathscr{Z}_{MM'} = \sum_{\{S_n\}\{S'_n\}} \prod_{\langle n,m\rangle} \exp\{&KB(S_n B_m) + K''B(S'_n S'_m)\\ &+ K(N-1)\, B(S_n S_m)\, B(S'_n S'_m) + K_0\}\end{aligned} \tag{51}$$

where $K_0 = NK + K''$. It can be rearranged as

$$\mathscr{Z}_{MM'} = \sum_{\{S_n\}\{S'_n\}} \prod_{\langle n,m\rangle} \exp\{KB(S_n S_m) + K_0\} \exp\{L_{nm} B(S'_n, S'_m)\} \tag{52}$$

with $L_{nm} = K'' + (N-1)\, KB(S_n S_m)$, which takes only two values:

$$[K'' + (N-1)K] \qquad \text{or} \qquad (K'' - K)$$

For each given $\{S_n\}$ spin configuration the sum over the spin configuration $\{S'_n\}$ represents the partition function of a N-state Potts model with inhomogeneous couplings $\{L_{nm}\}$: $\mathscr{Z}_{N\text{-Potts}}(L_{nm})$. Consequently we can write Eq. (51) as

$$\mathscr{Z}_{MM'} = \sum_{\{S_n\}} \prod_{\langle n,m\rangle} \exp\{KB(S_n, S_m) + K_0\}\, \mathscr{Z}_{N\text{-Potts}}(L_{nm}) \tag{53}$$

The strategy is to replace $\mathscr{Z}_{N\text{-Potts}}(L_{nm})$ by the corresponding quantity defined on the dual lattice and then reexpress $\mathscr{Z}_{MM'}$ now with two sets of spins, one on the lattice, the other being on the dual lattice. The Wu–Wang duality[17] relation is given by

$$\mathscr{Z}_{N\text{-Potts}}(L_{nm}) = N^{1-(MM')^*} \prod_{\langle n,m \rangle} \{e^{L_{nm}} - e^{-L_{nm}/(N-1)}\}$$
$$\times \prod_{\langle n,m \rangle^*} e^{L^*_{nm}/(N-1)} \mathscr{Z}_{N\text{-Potts}}(L^*_{nm}) \tag{54}$$

The first product is carried out over pairs of sites $\langle n, m \rangle$ and the second one over pairs of dual sites $\langle n, m \rangle^*$. Now let $(M, M')^*$ be the number of dual lattice sites and the relation between our notations and Wu–Wang notations is deduced from the expression of the corresponding U matrix for the N-state Potts model:

$$\mathscr{Z}^{\text{Wu–Wang}}_{N\text{-Potts}}(K^{\text{Wu–Wang}}_{nm}) = \prod_{\langle n,m \rangle} \exp\left(-\frac{L_{nm}}{N-1}\right) \mathscr{Z}_{N\text{-Potts}}(L_{nm}) \tag{55}$$

$$K^{\text{Wu–Wang}}_{nm} = \frac{N}{N-1} L_{nm} \tag{56}$$

or equivalently after exchanging L_{nm} and L^*_{nm}:

$$\mathscr{Z}_{N\text{-Potts}}(L_{nm}) =$$
$$\frac{1}{N^{1-MM'} \prod_{\langle n,m \rangle^*} \{e^{L^*_{nm}} - e^{-L^*/(N-1)}\} \prod_{\langle n,m \rangle} e^{L_{nm}/(N-1)}} \mathscr{Z}_{N\text{-Potts}}(L^*_{nm}) \tag{57}$$

Now we replace $\mathscr{Z}_{N\text{-Potts}}(L^*_{nm})$ by its definition as sum over all spin configurations $\{S'^*_n\}$ on dual lattice sites and with dual coupling L^*_{nm}:

$$\mathscr{Z}_{N\text{-Potts}}(L^*_{nm}) = \sum_{\{S'^*_h\}} \prod_{\langle n,m \rangle^*} \exp L^*_{nm} B(S'^*_n, S'^*_m) \tag{58}$$

Now with the identity:

$$\exp L^*_{nm} B(S'^*_n, S'^*_m) = \frac{1}{N} \{e^{L^*_{nm}} + (N-1) e^{-L^*_{nm}/(N-1)}\}$$
$$+ \frac{N-1}{N} \{e^{L^*_{nm}} - e^{-L^*_{nm}/(N-1)}\} B(S^*_h, S'^*_m) \tag{59}$$

we can reexpress the partition function of the N-state Potts model as

$$\mathscr{Z}_{N\text{-Potts}}(L_{nm}) = N^{1-(MM')^*} \sum_{\{S'^*_n\}} \prod_{\langle n,m \rangle^*} \left\{ \frac{e^{L^*_{nm}} + (N-1) e^{-L^*_{nm}/(N-1)}}{e^{L^*_{nm}} - e^{-L^*_{nm}/(N-1)}} \right.$$
$$\left. + (N-1) B(S'^*_n, S'^*_m) \right\} \prod_{\langle n,m \rangle} e^{-L_{nm}/(N-1)} \tag{60}$$

Using the duality relation for the coupling constant we arrive at an expression only function of L_{nm}, but summed over spins on dual lattice, i.e.,

$$\mathscr{Z}_{N\text{-Potts}}(L_{nm}) = N^{1-(MM')^*} \sum_{\{S_n'^*\}} \prod_{\langle n,m\rangle^*} \{e^{L_{nm}} + (N-1) e^{-L_{nm}/(N-1)} B(S_n'^*, S_m'^*)\} \tag{61}$$

We substitute this expression into Eq. (53) and obtain an expression for the partition function of the P_{NN} model:

$$\mathscr{Z}_{MM'} = \sum_{\{S_n\}\{S_n'^*\}} \prod_{\substack{\langle n,m\rangle \\ \langle n,m\rangle^*}} \left\{ \exp[NKB(S_n, S_m) + K''] + (N-1) \exp\left[-\frac{K''}{(N-1)} \right] B(S_n'^*, S_m'^*) \right\} e^{K_0} \tag{62}$$

This is a summation over the combined spin configurations $\{S_n\}$ and $\{S_n'^*\}$ which are grouped as in Fig. 2. A pair of spin S_n, S_m interacts thus with a pair of spin $S_n'^*$, $S_m'^*$, lying in a direction perpendicular to the direction of S_n, S_m with a 4-spin weight:

$$w(S_n, S_m; S_n'^*, S_m'^*) = \exp K_0 \left\{ \exp[NK\, B(S_n, S_m) + K''] + (N-1) \exp\left[-\frac{K''}{(N-1)} B(S_n'^*, S_m'^*) \right] \right\} \tag{63}$$

This is the generalization of Eq. (6) of Ref. 28 to $Z(N)$ spins. Depending on whether we consider vertical or horizontal links of the lattice we have the

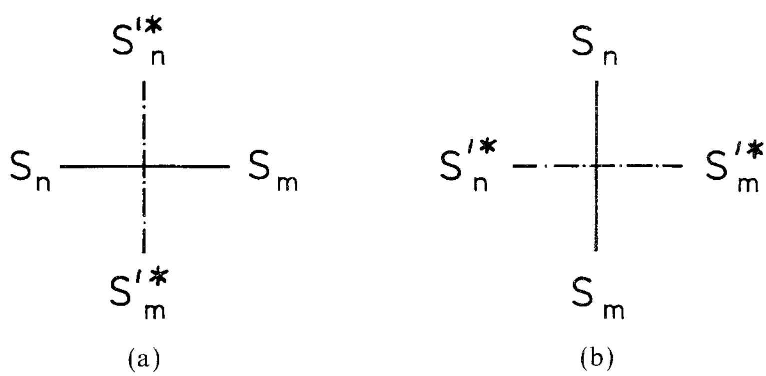


Fig. 2. Geometry of partial duality on S_n' spins.

two geometrical situations of Figs. 2a and 2b. We show that $w(S_n, S_m; \mathrm{S}_m'^*, S_m'^*)$ may be interpreted as weights of a $Z(N)$-vertex system defined on the medial lattice $\mathscr{L}_M$ [20] associated to the original square lattice. Let us consider Fig. 2a and define on the four links round a site of $\mathscr{L}_M$ the bond spins by

$$\begin{aligned} \sigma &= S_n S_n', \qquad & \tau &= S_n' S_m \\ \sigma' &= S_m' S_m, \qquad & \tau' &= S_n S_m' \end{aligned} \tag{64}$$

We next prove that they obey the constraint (see Fig. 3)

$$\sigma\sigma' = \tau\tau' \tag{65}$$

which may be interpreted as a $Z(N)$-vertex rule on $\mathscr{L}_M$ (generalizing the eight-vertex rule for which $N=2$). Since the bond variables $B(S_n, S_m)$ and $B(S_n'^*, S_m'^*)$ take each one only two values, $w(S_n, S_m; S_n'^*, S_m'^*)$ has only four values:

$$\begin{aligned}
a &= \{\exp K_0\}\left\{\exp(NK+K') + (N-1)\exp\left[-\frac{K'}{(N-1)}\right]\right\} = [1+(N-1)\omega] \\
c &= \{\exp K_0\}\left\{\exp(NK+K') - (N-1)\exp\left[-\frac{K'}{(N-1)}\right]\right\} = [1-\omega] \\
d &= \{\exp K_0\}\left\{\exp\left[-\frac{N}{(N-1)}K+K'\right] + (N-1)\exp\left[-\frac{K'}{(N-1)}\right]\right\} \\
&= [\omega'' + (N-1)\omega] \\
f &= \{\exp K_0\}\left\{\exp\left[-\frac{N}{(N-1)}K+K'\right] - \exp\left[-\frac{K'}{(N-1)}\right]\right\} = [\omega'' - \omega]
\end{aligned} \tag{66}$$

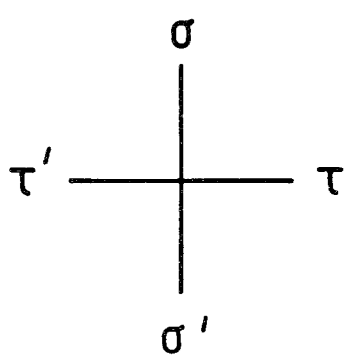


Fig. 3. Labeling of bonds round a vertex.

The weight a is obtained for $B(S_n, S_m) = B(S_n'^*, S_m'^*) = 1$ or when $S_n = S_m = \xi^1$ and $S_n'^* = S_m'^* = \xi^k$ with l, $k = 0, 1,..., (N-1)$. This choice implies that the bond spins are such that

$$\sigma = \sigma' = \tau = \tau' = \xi^p, \qquad p = l + k \qquad (\mathrm{mod}\ N) \tag{67}$$

and that there exists N vertex configurations shown in Fig. 4a, they can be deduced from one another by $Z(N)$ symmetry.

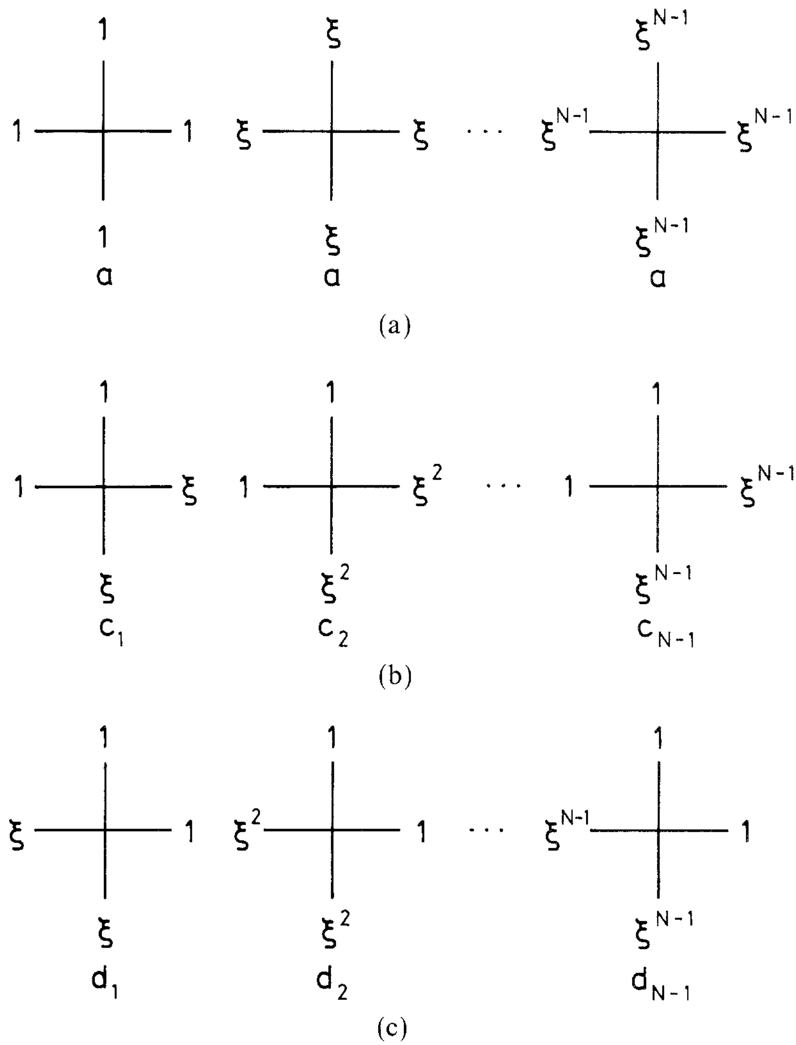
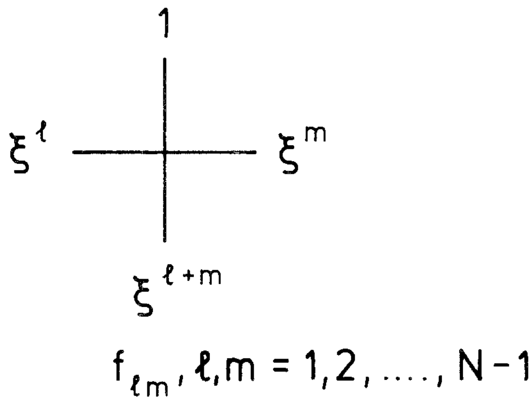
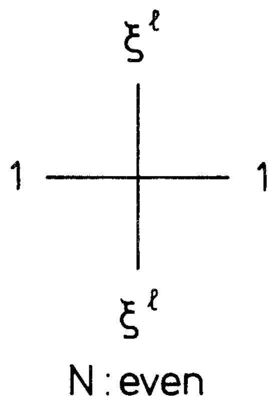


Fig. 4. Configurations of the $Z(N)$-symmetric vertex model and their respective weights.



(d)



(e)

Fig. 4 (*continued*)

The weight c corresponds to $B(S_n, S_m) = 1$ and $B(S_n'^*, S_m'^*) = (N-1)^{-1}$ or when

$$S_n = S_m = \xi^l, \qquad l = 0, 1, \ldots, (N-1)$$

$$S_n' = \xi^k \qquad \text{and} \qquad S_m' = \xi^{k'}, \qquad k = k'$$

$$k, k' = 0, 1, \ldots, (N-1) \tag{68}$$

consequently the bond spins are such that

$$\sigma = \tau' = \xi^{l+k} \qquad \text{and} \qquad \sigma' = \tau = \xi^{l+k'} \tag{69}$$

and fulfill Eq. (65). For chosen σ since $\sigma \neq \sigma'$ there are only $(N-1)$ possible values for σ', thus $N-1$ vertex configurations of the type shown in Fig. 4b. Letting σ run through $1, \xi, ..., \xi^{N-1}$ we generate all $(N-1)$ vertex configurations [by "$Z(N)$-spin-flip"] which belong to the same weight c.

At this point we notice that if we affect to each configuration of Fig. 4b a weight c_l, $l = 1, 2, ..., (N-1)$ and let it be the same for any other configuration deduced by $Z(N)$-spin-flip then these weights are those of a more general $Z(N)$-symmetric vertex model.

For weight d, it suffices to repeat the same argument after exchanging the role of $\{S_n\}$ and $\{S_n'^*\}$, or simply rotate the lattice by 90°. The configurations of the generalized $Z(N)$-vertex model are pictured in Fig. 4c with their weights d_l, $l = 1, 2, ..., (N-1)$.

Finally the f weight is related to the conditions $B(S_n, S_m) = B(S_n'^*, S_m'^*) = -(N-1)^{-1}$ or

$$\begin{aligned} &S_n = \xi^l \quad \text{and} \quad S_m = \xi^{l'}, \quad l \neq l', \quad l, l' = 0, 1, ..., N-1 \\ &S_n' = \xi^k \quad \text{and} \quad S_m' = \xi^{k'}, \quad k \neq k', \quad k, k' = 0, 1, ..., N-1 \end{aligned} \tag{70}$$

Again Eq. (65) is satisfied because

$$\sigma = \xi^{l+k}, \qquad \sigma' = \xi^{l'+k'}, \qquad \tau = \xi^{k+l'}, \qquad \tau' = \xi^{k'+l} \tag{71}$$

This leads to $N(N-1)^2$ configurations round a site of $\mathscr{L}_M$ and when N is even we also could have configurations with two values of bond spins crossing each other (see Fig. 4d). In the generalized $Z(N)$-vertex model we define $(N-1)^2$ weights f_{lm} as shown in Fig. 4e, any other configuration deduced by $Z(N)$-symmetry having then the same weight f_{lm}.

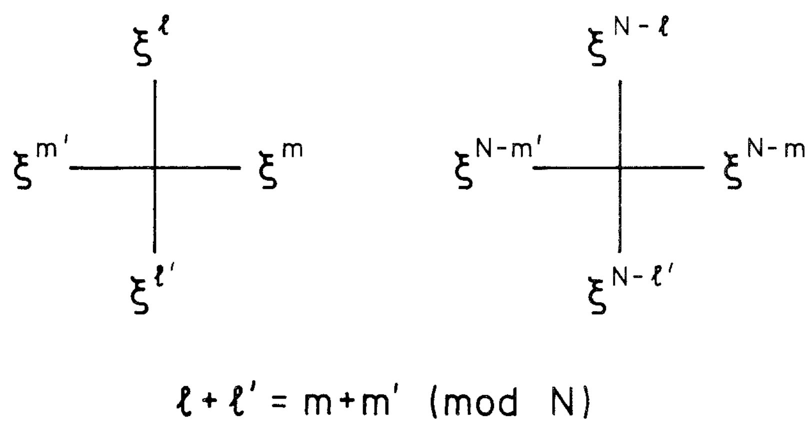


Fig. 5. Complex conjugation $\mathscr{C}$ of configurations.

Because the values of the bond spins are unimodular numbers for $N > 2$, the vertex configurations can be related pairwise by complex conjugation: $\mathscr{C}$ which maps c_l to c_{N-l}, d_m to d_{N-m} and $f_{lm} \to f_{N-l,m} \to f_{l,N-m} \to f_{N-l,N-m}$ (see Fig. 5). When N is even there is one configuration weight besides a which maps into itself. We shall see the impact of $\mathscr{C}$ later on the symmetries of the model.

In conclusion, we have obtained a new $Z(N)$-symmetric vertex system on $\mathscr{L}_M$ having N^3 configurations, coming from N^4 spin configurations of the original lattice, and depending on N^2 weights: a, c_l, d_m, f_{lm} with $l, m = 1, 2,..., N-1$. Moreover the P_{NN} model is found to be equivalent (up to boundary conditions) to a staggered *restricted* $Z(N)$-vertex model on a sqare lattice whereby $c_l = c$, $d_m = d$, $f_{lm} = f$. Our $Z(N)$-vertex system defined by Eq. (65) differs from the generalization of the eight-vertex model proposed by Belavin, because he assumes another constraint on the bond spins, namely,

$$\sigma\tau = \sigma'\tau' \tag{72}$$

It is rather a generalization of the "N-color nonintersecting string model" of Schultz and Perk.[18,19]

5. PROPERTIES OF THE $Z(N)$-VERTEX MODEL

The partition function of this $Z(N)$-vertex model, defined on a square lattice (M columns and M' rows) with periodic boundary conditions is given by

$$\mathscr{Z}_{MM'}(a, c_l, d_m, f_{lm}) = \sum_{\{\text{conf}\}} a^{N(a)} c_l^{N(c_l)} d_m^{N(d_m)} f_{lm}^{N(f_{lm})} \tag{73}$$

where $N(a)$, $N(c_l)$, $N(d_m)$, and $N(f_{lm})$ are the number of vertices of type a, c_l, d_m, f_{lm} appearing in a given lattice configuration, and the summation is carried over all allowable configurations. In this section, we are concerned with the symmetry properties of this partition function.

First, we observe that Eq. (65) goes into itself under a clockwise rotation of 90° of the lattice. Simple inspection shows that this rotation S, as defined in Section 3 performs now the mapping

$$S\colon (a, c_l, d_m, f_{lm}) \to (a, d_{N-l}, c_m, f_{m,N-l}) \tag{74}$$

$$\mathscr{Z}_{MM'}(a, c_l, d_m, f_{lm}) = \mathscr{Z}_{M'M}(a, d_{N-l}, c_m, f_{m,N-l}) \tag{75}$$

However, only S^4 is equal to the identity and both S^2 and S^3 are still nontrivial and generate two more relations similar to Eq. (75). For the restricted $Z(N)$-vertex model, S becomes in fact an involution.

Second, Eq. (65) will not be altered, if we multiply the bond spin τ (resp. σ) by ξ^p and the bond spin τ' (resp. σ') by ξ^{N-p}, with $p = 1, 2,..., N-1$. So "flipping" the bond spins consistently along an axis would generate another $Z(N)$-vertex system. But since $\xi^p \neq \xi^{N-p}$ for $N > 2$, the original homogeneous vertex system is now converted into an inhomogeneous system of vertices alternating along an axis, or even staggered if we alternate the spin-flipping operation from row to row, or from column to column.

Third, Eq. (65) retains its form again if we multiply σ or τ (or resp. σ' and τ') by a same number ξ^p, $p = 1, 2,..., N-1$. If we carry out this operation for bond-spins located on parallel directions to a diagonal of the lattice, we may hope that the system has a "stairways symmetry," discovered by Fan and Wu[29] for the eight-vertex model. This is not so here because multiplying σ and τ by ξ^p generates a $Z(N)$-vertex system different from the one obtained by multiplying σ' and τ' by ξ^p, except for $N = 2$.

There is however a pseudo-weak-graph symmetry[29] D, which exists for $N > 2$ but is not an involution and satisfies $D^4 = I$. It reduces only to an involution under some restrictions, as we shall see later on.

The starting point is a representation of the statistical weight associated to a vertex configuration of Fig. 4a–e subject to Eq. (65). Following Baxter[20] we denote this weight by $w(\tau', \sigma'; \sigma, \tau)$ which should take the values: a, c_l, d_m, f_{lm} for various allowable configurations of σ, σ', τ, τ' round a vertex. Making use of Eq. (6) we can write

$$w(\tau', \sigma'; \sigma, \tau) = N^{-2} \left\{ a^* \sum_{k=0}^{N-1} \sigma^k \sigma'^k \tau^{N-k} \tau'^{N-k} \right.$$

$$+ \sum_{k=0}^{N-1} \sum_{l=1}^{N-1} c_l^* \sigma^{k+l} \sigma'^1 \tau^{N-1} \tau'^{2N-l-k} + \sum_{k=0}^{N-1} \sum_{m=1}^{N-1} d_m^* \sigma^{m+k} \sigma'^k \tau^{2N-m-k} \tau' N - k$$

$$\left. + \sum_{k=0}^{N-1} \sum_{l,m=1}^{N-1} f_{lm}^* \sigma^{k+l+m} \sigma'^k \tau^{2N-m-k} \tau'^{2N-k-l} \right\} \tag{76}$$

where a^*, c_l^*, d_m^*, f_{lm}^* describe the original weights through the linear relations:

$$a = N^{-1} \left\{ a^* + \sum_{l=1}^{N-1} c_l^* + \sum_{m=1}^{N-1} d_m^* + \sum_{l=1}^{N-1} \sum_{m=1}^{N-1} f_{lm}^* \right\} \tag{77a}$$

$$c_{l'} = N^{-1} \left\{ a^* + \sum_{l=1}^{N-1} c_l^* + \sum_{m=1}^{N-1} \xi^{l'm} d_m^* + \sum_{l=1}^{N-1} \sum_{m=1}^{N-1} \xi^{l'm} f_{lm}^* \right\} \tag{77b}$$

$$d_{m'} = N^{-1}\left\{a^* + \sum_{l=1}^{N-1} \xi^{lm'} c_l^* + \sum_{m=1}^{N-1} d_m^* + \sum_{l=1}^{N-1}\sum_{m=1}^{N-1} \xi^{mm'} f_{lm}^*\right\} \tag{77c}$$

$$f_{l'm'} = N^{-1}\left\{a^* + \sum_{l=1}^{N-1} \xi^{lm'} c_l^* + \sum_{m=1}^{N-1} \xi^{ml'} d_m^* + \sum_{l=1}^{N-1}\sum_{m=1}^{N-1} \xi^{lm'+ml'} f_{lm}^*\right\} \tag{77d}$$

Equation (76) can be considered as the sum N^3 terms, i.e.,

$$w(\tau', \sigma'; \sigma, \tau) = \sum_{k=1}^{N^3} w^{(k)}(\tau', \sigma'; \sigma, \tau) \tag{78}$$

each of which is of the generic form: $g^*\sigma^l\sigma'^m\tau^{N-l'}\tau'N-m'$, where g^* may be any of the a^*, c_l^*, d_m^*, f_{lm}^*. To this term we associate the vertex configuration of Fig. 6 where $l+m=l'+m'$ (mod N). This correspondence is then one to one with all the configurations of Fig. 4. Consequently we see that the $\mathscr{C}$ transformation performs the mapping:

$$\mathscr{C}\colon\ g^*\sigma^l\sigma'^m\tau^{N-l'}\tau'N-m' \to g^*_{\mathscr{C}}\sigma^{N-l}\sigma'N-m\tau^{l'}\tau'^{m'}$$

Let us label each site of the square lattice by (i, j) where $i = 1, 2,..., M$ and $j = 1, 2,..., M'$. Thus at a vertex (i, j) the statistical weight is a function of τ_{ij}, σ_{ij}, $\sigma_{i+1,j}$, and $\tau_{i,j+1}$, namely,

$$w_{ij} = w(\tau_{ij}, \sigma_{ij}; \sigma_{i+1,j}, \tau_{i,j+1}) \tag{79}$$

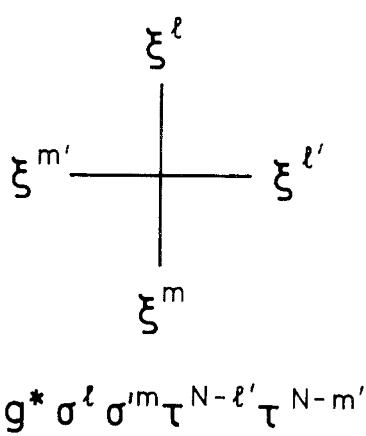


Fig. 6. Weak-graph symmetry correspondence between term $g^*\sigma^l\sigma'^m\tau^{N-l'}\tau^{N-m'}$ and vertex configuration.

The partition function of the system can now be written as a summation over all bond spins:

$$\mathscr{Z}_{MM'}(a, c_l, d_m, f_{lm}) = \sum_{\{\sigma_{ij}, \tau_{ij}\}} w_{11} w_{12} \cdots w_{MM} \tag{80}$$

Upon substitution of Eq. (78) into the right-hand side of Eq. (80), we obtain a sum of $N^{3MM'}$ terms of the form $w_{11}^{(k_{11})} w_{12}^{(k_{12})} \cdots w_{MM'}^{(k_{MM'})}$, each k_{ij} runs from 1 to N^3; now each of such term represents an assignment of allowable vertex configurations on the MM' sites of the lattice, due to the correspondence defined after Eq. (78). On each bond, for example the one connecting vertex (i, j) to vertex $(i-1, j)$, appears a product of $\sigma_{ij}^p \sigma_{ij}^q$, p from vertex (i, j) and q from vertex $(i-1, j)$ as shown in Fig. 7, in the expansion of the right-hand side of Eq. (80). We have to sum over all values of bond spins of the lattice, in particular over all $Z(N)$ values of σ_{ij}. The contribution of this summation is only nonzero if and only if $p+q=0 \pmod N$, in other words if the vertex configuration at vertex (i, j) and the vertex configuration at vertex $(i-1, j)$ display the same value of the bond spin σ_{ij}; note that one of the configurations is of the conjugate type so that $q = N - p$ essentially. Repeating the same argument for all bond spins of the lattice we find that our original covering of the lattice by vertex configurations of weights a, c_l, d_m, f_{lm} may be replaced by configurations of another $Z(N)$-vertex system with weights a^*, c_l^*, d_m^*, and f_{lm}^*, i.e.,

$$\mathscr{Z}_{MM'}(a, c_l, d_m, f_{lm}) = \mathscr{Z}_{MM'}(a^*, c_l^*, d_m^*, f_{lm}^*) \tag{81}$$

Equation (77) defines a transformation D which is a complex linear transformation of the weights:

$$D\colon (a, c_l, d_m, f_{lm}) \to (a, c_l, d_m, f_{lm}) \tag{82}$$

Only the fourth power of D is equal to the identity, thus D^2 and D^3 are still nontrivial. To prove this it is sufficient to iterate Eq. (77), a simple but cumbersome procedure which shall not be presented here.

The conditions under which Eq. (77) define a real involution is

$$\begin{gathered} c_l = c_{N-l}, \qquad d_m = d_{N-m} \\ f_{lm} = f_{N-1,m} = f_{l,N-m} = f_{N-l,N-l} \end{gathered} \tag{83}$$

which mean that the $Z(N)$-vertex configurations are now invariant under $\mathscr{C}$. There are then only $(p+1)^2$ [resp. $(p+2)^2$] real weights for $N = 2p+1$ (resp. $N = 2p+2$) and Eq. (77) reduce now to, for $N = 2p+1$,

$$a = \frac{1}{2p+1}\left\{a^* + 2\sum_{l=1}^{p} c_l^* + 2\sum_{m=1}^{p} d_m^* + 4\sum_{m,l=1}^{p} f_{lm}^*\right\}$$

$$c_{l'} = \frac{1}{2p+1}\left\{a^* + 2\sum_{l=1}^{p} c_l^* + 2\sum_{m=1}^{p} \cos\frac{2\pi}{2p+1} ml' d_m^* \right.$$
$$\left. + 4\sum_{l,m=1}^{p} \cos\frac{2\pi}{2p+1} ml' f_{lm}^*\right\}$$

$$d_{m'} = \frac{1}{2p+1}\left\{a^* + 2\sum_{l=1}^{p} \cos\frac{2\pi}{2p+1}\, lm' c_l^* + 2\sum_{m=1}^{p} d_m^* \right.$$
$$\left. + 4\sum_{l,m=1}^{p} \cos\frac{2\pi}{2p+1} mm' f_{lm}^*\right\}$$

$$f_{l'm'} = \frac{1}{2p+1}\left\{a^* + 2\sum_{l=1}^{p} \cos\frac{2\pi}{2p+1} lm' c_l^* + 2\sum_{m=1}^{p} \cos\frac{2\pi}{2p+1} l'm\, d_m^* \right.$$
$$\left. + 4\sum_{l,m=1}^{p} \cos\frac{2\pi}{2p+1} ml' \cos\frac{2\pi}{2p+1} m'l f_{lm}^*\right\}$$
$$\text{for} \quad l', m' = 1, 2,..., N-1 \tag{84}$$

For $N = 2p+2$ these equations become a bit more involved due to the presence of the weights c_{p-1}, d_{p-1}, $f_{p-1,m}$, $f_{l,p-1}$, $f_{p-1,p-1}$.

We observe that for $N > 3$, there are no set of nontrivial weights invariant under D, although $D^2 = I$. To get a nontrivial set of invariant weights one should enlarge the symmetry of the weights by requiring that

$$\begin{aligned} c_l &= c && \text{for all} \quad l' = 1, 2,..., N-1 \\ d_m &= d && \text{for all} \quad m' = 1, 2,..., N-1 \\ f_{lm} &= f && \text{for all} \quad l', m' = 1, 2,..., N-1 \end{aligned} \tag{85}$$

But these conditions describe precisely our restricted $Z(N)$-vertex model which has emerged from the partial-duality transformation of the P_{NN} model. The D transform of the weights has now the simple form, for all N:

$$\begin{aligned} a &= N^{-1}\{a^* + (N-1)c^* + (N-1)d^* + (N-1)^2 f^*\} \\ c &= N^{-1}\{a^* + (N-1)c^* - d^* + (N-1)f^*\} \\ d &= N^{-1}\{a^* - c^* + (N-1)d^* + (N-1)f^*\} \\ f &= N^{-1}\{a^* - c^* - d^* + f^*\} \end{aligned} \tag{86}$$

Note that for $N=2$ we recover the formulas for the eight-vertex model after a reconversion of the weights to standard ones. The associated invariant manifold is given by the linear equation:

$$a = c + d + (N-1) f \tag{87}$$

It is of course symmetric in c and d, this fact reflects the invariance of the restricted $Z(N)$-vertex model under 90° rotation of the lattice. Condition of self-duality under weak-graph symmetry of Eq. (87) is, using Eq. (66) simply $\omega = \omega''$ for the P_{NN} model.

All the symmetries considered here can be studied also in the $Z(N)$-belavin model, which is not invariant under 90° rotation of the lattice; but there is a restricted $Z(N)$-Belavin model, which exhibits spin-flip invariance along an axis and has a self-dual manifold of weights under weak-graph symmetry similar to Eq. (87).[30] At $N=2$, i.e., the eight-vertex model all the symmetries of the $Z(N)$-Belavin model coincide with those studied here, this is because Eqs. (65) and (72) are then the same condition.

Formulas (86) may be used to map a restricted $Z(N)$-vertex model to a Schultz–Perk model, where the f weight is zero. The situation is akin to that of a critical eight-vertex model which is mapped to a six-vertex model in the disorder phase through a weak-graph symmetry transformation. At this point it might be useful to point out that by inverting Eq. (86) and using Eq. (66) we obtain the vertex weights in terms of ω and ω'':

$$\begin{aligned} a^* &= 1 - (N-1)\omega'' \\ c^* &= 1 - \omega'' \\ d^* &= N\omega \qquad \text{and} \qquad f^* = 0 \end{aligned} \tag{88}$$

The dual invariant Δ_N of Section 3 appears as

$$\Delta_N = (N-1)\frac{(c^*)^2 + (d^*)^2 - (a^*)^2}{Nc^*d^*} - (N-2)\frac{(c^* - a^*)(d^* - a^*)}{Nc^*d^*} \tag{89}$$

which is simply a generalization of the Lieb invariant of a six-vertex model; upon comparing Eq. (30) to Eq. (89) we now understand the meaning of the variables **a** and **b** introduced before.

Finally, since the P_{NN} model is equivalent to a staggered restricted $Z(N)$-vertex model, whereby weights c^* and d^* are exchanged on the two interpenetrating sublattices, which possess incidentally the same Δ_N, the staggering effect disappears if $c^* = d^*$. This happens precisely when the self-duality condition of Section 3 is satisfied. However on this self-dual line, although we have a homogeneous Schultz–Perk vertex system, the weights do not satisfy the integrability conditions,[18] except for $N=2$ or for $\omega=\omega''$, this later case is treated in the next section on an anisotropic square lattice.

6. THE N^2-STATE POTS MODEL

In this section we consider the particular staggered vertex system representing an anisotropic N^2-state Potts model. From Section 3 we see that $\Delta_{N,v}=\Delta_{N,h}=-1$. The weights of the two interpenetrating sublattices are arranged as follows:

$$\begin{aligned} a_h &= 1+(N-1)\omega_h, \qquad c_h=1-\omega_h, \qquad d_h=N\omega_h, \qquad f_h=0 \\ a_v &= 1+(N+1)\omega_v, \qquad c_v=N\omega_v, \qquad d_v=1-\omega_v, \qquad f_v=0 \end{aligned} \tag{90}$$

We observe the disappearance of vertex configurations of weight f_v (resp. f_h), leaving only one color vertices of weight a and two color vertices of weight c and d, which are staggered. Moreover, as in the six-vertex model we find

$$\Delta_{N,j}=-1 \Rightarrow a_j=c_j+d_j, \qquad j=v,h \tag{91}$$

So, analogously we introduce the "staggered spectral" parameters:

$$\alpha=\frac{1-\omega_h}{N\omega_h} \qquad \text{and} \qquad \alpha'=\frac{N\omega_v}{1-\omega_v} \tag{92}$$

of a canonical representation; after proper global normalization by $\{N\omega_h(1-\omega_v)\}^{-1}$ we obtain

$$\begin{aligned} a_h &= 1+\alpha, \qquad c_h=\alpha, \qquad d_h=1, \qquad f_h=0 \\ a_v &= 1+\alpha', \qquad c_v=\alpha', \qquad d_v=1, \qquad f_v=0 \end{aligned}$$

At self-duality or non-staggering limit we have

$$\alpha=\alpha' \qquad \text{or} \qquad (1-\omega_h)(1-\omega_v)=N^2\omega_v\omega_h \tag{93}$$

This is precisely the model of Schultz called nonintersecting string model in Ref. 18. On a square lattice it may be described by the totality of N-color nonintersecting polygonal contours, because we have now a homogeneous system of N-color polygon corners of weights $c=\alpha$ and $d=1$, in which only bonds of same color can cross each other with a weight $a=1+\alpha$. It is known that the model is soluble and that its partition function obtained through the inverse relation is eaxtly that of a six-vertex model.[18]

To clarify this last observation we recall that the N^2-state Potts model is in fact also equivalent to a staggered-six-vertex model, with weights on the sublattices (see Ref. 31):

$$\begin{aligned} &a_h = 1, \quad && b_h = x_h, \quad && c_h = 1 + x_h e^{\lambda}, \quad && c'_h = 1 + x_h e^{-\lambda} \\ &a_v = x_v, \quad && b_v = 1, \quad && c_v = x_v + e^{\lambda}, \quad && c'_v = x_v + e^{-\lambda} \end{aligned} \tag{94}$$

The six-vertex model has a canonical parametrization with Lieb's invariant:

$$\Delta_h = \Delta_v = -\cosh\lambda = -\frac{(N^2)^{1/2}}{2} \tag{95}$$

and "staggered spectral" parameters α and α' defined by

$$\begin{aligned} x_h &= \frac{\sinh\alpha}{\sinh(\lambda-\alpha)} = \frac{e^{K_h}-1}{(N^2)^{1/2}} \\ x_v &= \frac{\sinh(\lambda-\alpha')}{\sinh\alpha'} = \frac{e^{K_v}-1}{(N^2)^{1/2}} \end{aligned} \tag{96}$$

It is known[2,32] that the model is soluble for

$$\alpha = \alpha' \pm ni\frac{\pi}{2} \qquad (n, \bmod 2) \tag{97}$$

For $n=0$ we recover the nonstaggering case (or self-duality) and $n=1$ corresponds to the antiferromagnetic critical regime of Baxter.[2] It is now clear that (at self-duality) the partition function of the Schultz–Perk model coincides precisely with that of the six-vertex model, and this surprising fact occurs whenever the number of components of the Potts model is an exact square where two vertex representations of the Potts model coexist (incidentally this was known already for the 4-state Potts model).

7. SOME EXTENTIONS AND CONCLUSIONS

Before closing this study we would like to bring up a straightforward generalization of the P_{NN} model, which consists of breaking the $Z(N^2)$-symmetry down to the $Z(N)\otimes Z(N)$ symmetry. For simplicity we assume the lattice to be isotropic and consider instead of Eq. (5) a bond energy depending on three coupling constants K, K', and K'':

$$\beta H_{nm} = K_0 + KB(S_n, S_m) + K'B(S'_n, S'_m) + (N-1)\,K'B(S_n, S_m)\,B(S'_n, S'_m) \tag{98}$$

For $N=2$ we recover the Ashkin–Teller model on an isotropic lattice. Partial duality on the spins S'_n can still be performed, and a staggered $Z(N)$-vertex system emerges with the weights:

$$\begin{aligned}
a &= \exp\{K+K'+(N-1)K''\} - (N-1)\exp\{K-(N-1)^{-1}K'-K''\} \\
c &= \exp\{K+K'+(N-1)K''\} - \exp\{K-(N-1)^{-1}K'-K''\} \\
d &= \exp\{-(N-1)^{-1}K+K'-K''\} - (N-1)\exp\{[-K-K'+K''](N-1)^{-1}\} \\
f &= \exp\{-(N-1)^{-1}K+K'-K''\} - \exp\{[-K-K'+K''](N-1)^{-1}\}
\end{aligned} \tag{99}$$

The staggering effect lies in the exchange of weights c and d on the two interpenetrating sublattices and disappears for $d=c$ or

$$\begin{aligned}
\exp\{K+K'+NK'\} &= \exp\{K-(N-1)^{-1}K'\} + \exp\{-(N-1)^{-1}K+K'\} \\
&\quad + (N-1)\exp\{-(N-1)^{-1}(K+K'-NK'')\}
\end{aligned}$$

which is symmetric in K and K'.

Now it is instructive to look at some special cases. Besides the P_{NN} model, for which $K=K'$, we have two other N^2-state spin system of interest.

If $K=K'$, we will have two interacting N-state Potts models but nonequivalent to the P_{NN} model for $N>2$, because it has only a $Z(N)\otimes Z(N)$ symmetry and has a decoupling limit for $K''=0$ where it reduces itself to two noninteracting N-state Potts models.

Let us set $K'=K''$ and recall it K, simultaneously we rename K by K''. This is still a P_{NN} model whose equivalent staggered vertex model has precisely the weights given by Eq. (88). This fact confirms the consistency of the weak-graph symmetry given by Eq. (86) and generalizes a situation well known in the Ashkin–Teller model.

By setting $K'=K''=0$ in Eq. (99), we obtain the representation of the N-state Potts model by a special staggered Schultz–Perk vertex system having only two non-zero weights:

$$a=Ne^K, \qquad c=0, \qquad d=N\exp[-(N-1)^{-1}K] \tag{100}$$

The weights c and d alternate on the sublattices which form the medial lattice of the original lattice. Close inspection shows that this is a representation proposed long ago by Wu and Lin[22] for the Ising model and extended to other spin value by Wu.[33] One could have equally set $K=K''=0$ and obtain another representation of the N-state Potts model by a different staggered Schultz–Perk vertex model, but this vertex model is related to the one defined by Eq. (100) by weak-graph symmetry, or equivalently by spin duality [see Eq. (20)].

It is hoped that this study serves as an introduction to future investigations on $Z(N)$-spin systems in two dimensions which depend on two parameters. We have not touched here the critical behavior of the system. Since the self-dual line is not exactly soluble for $N > 2$, little can be said about the nature of the transition along the self-dual line or across the two other presumed critical lines bifurcating from the multicritical point I_3 in Fig. 1. It is perhaps useful to await for results from numerical simulations to probe deeper into the structure of the system. Nevertheless one may try to seek an equivalence with a generalized Coulomb gas as done by L. P. Kadanoff[34] for the Ashkin–Teller model and Potts model and more recently by B. Nienhuis for other systems,[35] to extract some more information. On the other hand the new $Z(N)$-symmetric vertex system is also of interest. One may search for integrable subfamilies which in some appropriate limit coincide with the integrable families found by Schultz and Perk.[18] One may eventually seek a spin representation of these $Z(N)$-vertex model which would generalize the Kadanoff–Wegner Ising spin representation of the eight-vertex model.[36]

To summarize, in this paper we have basically brought about some new aspects of a class of $Z(N^2)$-spin system depending on two parameters probond. We have found its equivalent vertex model and studied an even more general $Z(N)$-vertex model which displays a pseudo weak-graph symmetry. Some by-products are the representations of the N^2-state and N-state Potts models by two special Schultz–Perk vertex systems which shed light upon many connections among some statistical systems in two dimensions. Last but not least, we should point out that the bulk of these results can only be obtained because of the $Z(N^2)$ symmetry assumed from the beginning for the spin system.

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